expressed relative to a variety of internal reference-frames; some of them only indirectly related to the CNS. In terms inherent to the 'body', the ordered set of quantities may be the two angles at the shoulder and at the elbow, or the contraction produced by the flexor and extensor muscles, or, in terms of the CNS, the sums of firing frequencies of the motoneurons innervating them. It must be emphasized that the difference between the two kinds of reference-frames is that one set is external (extrinsic) to the CNS, in the sense that it exists independently of the CNS, while the other kinds of reference-frames are dependent on, and reflect the properties and constraints of, the CNS.

Given the above and given that a tensor is a reference-frame invariant vector-relationship (cf. HOFFMANN, 1966), *the limb movement is, by definition, a tensorial entity.* Thus, as shown in Fig. 1, the limb displacement can be characterized by the movement expressed as a vector (tensor of rank one) in both the external space and the internal hyperspace, and the different vectorial expressions can be transformed from one to the other by the transformation matrix of $\overline{\Lambda}^{\mu}_{i}$. As KRON (1939) pointed out in his epoch-making tensor-theory of electrical networks, the entirety of the transformation-matrices establishing relations among expressions in different reference-frames of a physical vector is one 'geometrical object', where the particular matrices are different expressions of one tensor.*

Thus, in tensorial terms the implementation of a straight upward limb-displacement is seen as follows. The intended movement vector $\overline{U}$ is characterized as a sum of a y and an x-component, relative to the external frame of reference. However, in order to execute the displacement, the vector must *of necessity* be present internally. An ordered set of two quantities of α and β has to be established, via the transformation by $\overline{\Lambda}$. The reference-frame invariance of movement-vectors can be used to introduce a formal geometrical definition of coordination: *Motor coordination is a geometrical transformation of an intended movement-vector into an executable expression of the same vector.* Although both vectors are expressed in the CNS the intended vector *specifies* the external goal vector *measured* in bodily terms, whereas the execution vector *generates* the movement.

While in the above case the needed α, β internal components can be directly transformed from the external x, y components through the limb-movement tensor, $\overline{\Lambda}$, one has to point out a fundamental problem in the general case. If the internal spaces employed by the body and CNS have more dimension than the external, the execution of, e.g., a *two*-dimensional movement) by an arm with *three* degrees of freedom) is theoretically not unique. In fact, the number of possibilities of decomposing a *two*-dimensional vector into *three* components is infinite. Still, in practice particular individual solutions do occur every time the body moves. This paper provides a hypothetical scheme by which this uniqueness problem of the coordination could be solved by virtue of a cerebellar metric tensor.

Quantitative expressions of a limb-movement tensor

Let us provide a simple quantitative example of limb movements as tensors. In general, the limb-movement tensor determines the relationship between different expressions of the limb-displacement vector $\overline{U}$. The general expression of an infinitesimal $\overline{U}$ by μ infinitesimal components may be

$$\overline{U}^{\mu}, \text{ e.g. } \overline{U} = \overline{U}(\alpha, \beta).$$

While $\overline{U}$ can be expressed in both the intrinsic and extrinsic frames of reference:

$$\overline{U} = \overline{U}(x, y) \quad \text{or} \quad \overline{U} = \overline{U}(R, \Psi) \quad \text{or} \quad \overline{U} = \overline{U}(\alpha, \beta)$$

there *is* a relationship among all of these, expressing the *same* geometrical object. In tensorial terms, and using the Einstein summation convention

$$\overline{U}^{\mu} = \overline{\Lambda}^{\mu}_{i} \cdot \overline{U}^{i}.$$

For example, selecting a polar system of coordinates *as external reference frame*, and α, β as a *body-reference-frame*, the expression becomes:

$$\overline{U}^{\mu}(\alpha, \beta) = \overline{\Lambda}^{\mu}_{i} \cdot \overline{U}^{i}(R, \Psi).$$

In this expression the $\overline{\Lambda}^{\mu}_{i}$ limb-movement tensor has the following four particular components:

$$\overline{\overline{\Lambda}}{}^{\mu}_{i} = \frac{\partial \overline{U}^{\mu}}{\partial \overline{U}^{i}} = \begin{pmatrix} \dfrac{\partial \alpha}{\partial R} & \dfrac{\partial \alpha}{\partial \Psi} \\ & \\ \dfrac{\partial \beta}{\partial R} & \dfrac{\partial \beta}{\partial \Psi} \end{pmatrix}$$

These particular values can be established, if

$$\alpha = \Psi - \arccos \frac{R}{2r}$$

(see Fig. 1)

$$\beta = \pi - 2 \arcsin \frac{R}{2r}$$

Therefore, after the above partial differentiation the limb-movement tensor is

$$\overline{\overline{\Lambda}}{}^{\mu}_{i} = \frac{\partial \overline{U}^{\mu}}{\partial \overline{U}^{i}} = \begin{pmatrix} \dfrac{\partial \alpha}{\partial R} & \dfrac{\partial \alpha}{\partial \Psi} \\ & \\ \dfrac{\partial \beta}{\partial R} & \dfrac{\partial \beta}{\partial \Psi} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2r} \cdot \dfrac{1}{\sqrt{1 - \dfrac{R^2}{4r^2}}} & 1 \\ & \\ \dfrac{-1}{r} \cdot \dfrac{1}{\sqrt{1 - \dfrac{R^2}{4r^2}}} & 0 \end{pmatrix}$$

* It seems necessary to point out that a tensor need not be a square matrix; differential geometry routinely utilizes rectangular projection tensors: e.g. two-by-three matrices for embedding a two-dimensional surface into a three-dimensional space (cf. COBURN, 1955, p. 216).