of performing the same numerical transformation can be summarized by a matrix as illustrated in Fig. 2B. Such an array of quantities is an *arithmetical abstraction* of the function of particular networks shown in Fig. 2A. Expressed only by an arithmetic formula of a matrix, the function of a network appears to be merely to transform an input vector into an output vector, where both vectors are given in some (unknown) particular frame of reference.

If one knew the coordinate axes, such vector-transformations could be expressed at a higher level of abstraction in a more general form which applies to a set of frames of reference where the angle A between the axes is variable. Such an *algebraic expression* of the transformation-matrix, as a function of A, is shown in Fig. 2C. *Because there is no reason to assume that the CNS is limited to orthogonal coordinate systems, general (non-orthogonal, i.e. oblique) frames of reference are used throughout Tensor Network Theory.*

The algebraic expression in Fig. 2C defines the transformation-matrix implemented by the networks shown in Figs 2A, B in a relatively general manner. First, by selecting a particular value for the variable A, a concrete numerical matrix is specified. Then, there are an infinite number of network implementations of a simple particular matrix. Still, neither the arithmetical, nor the algebraical expression of a matrix reveals the functional advantage gained from the vectorial transformation implemented by the matrix. In short, the numbers in the matrices do not explain why a transformation is necessary.

The functional meaning of the transformation through all networks in Figs 2A, B, C becomes evident by a visualization of the vector transformation (Fig. 2D). It is intuitively clear there, that the input and output of the matrix transformation are two different kinds of vectorial expressions both assigned to one and the same physical location P, an invariant. The components v_i of the input vector are *covariant* (they are obtained by the orthogonal projection method) while the components v^j of the output vector are *contravariant* (obtained by the parallelogram method). *The differences between co- and contravariant vectors (and their tensorial relation through the metric) are of cardinal importance in the Tensor Network Theory of CNS.* Since the present work is built upon this conceptual foundation, it must be clearly understood before proceeding (cf. ref.[21]). The significance of defining the covariant or contravariant nature of a vector when using a non-orthogonal coordinate system is illustrated in Fig. 2D. Note that not only the numerical values of the components are different in each version of the vector, but they also have visibly different features. For instance, the covariant components can be established independently of one another, but they do not add up to the invariant. These features are opposite to those of the contravariants. These profound differences provided the basis for us to postulate[21,22] that sensory systems in the CNS are using expressions of covariant type while motor systems use contravariant-type components. These two vectorial versions warrant an important distinction. *Indeed, because the CNS need not be limited to the use of orthogonal systems, it is meaningless to introduce vectorial notation into neuroscience without specifically characterizing the components of each vector as covariant or contravariant, or else proving that the implied frame of reference is indeed orthogonal.* It is apparent that the transformation from v_i to v^j in Fig. 2D expresses a general relationship that exists not only for the specifically depicted v_i and v^j, but for all co- and contravariant pairs of vectors. Such a relation exists in every frame of reference, regardless of the directions or number of axes in the coordinate system.

The covariant-contravariant vectorial relationship may be expressed in a totally coordinate-system-free general manner by the mathematical device g^{ij}, the so-called metric tensor. The metric tensor transforms a covariant vector into its contravariant counterpart. This operation does not only occur in the two dimensional space illustrated in Fig. 2D, but may occur in any n-dimensional hyperspace. In fact, we use two-dimensional Euclidean geometry here only to facilitate the visualization of the vectorial relations that can exist in multidimensional, non-Euclidean CNS hyperspaces, where not rectilinear (or even not linear) non-orthogonal reference frames may be used.

The contravariant metric tensor is formally expressed in tensor analysis notation in Fig. 2E. This expression, while not intuitively obvious, is immensely powerful; it is devoid of the limitations of using particular reference frames. The coordinate-system-free tensor notation g^{ij} encompasses, in a generalized form, all particular expressions given in any frame of reference. Thus, tensor analysis appears to be the most appropriate language to describe CNS functions, since it can deal, in a general quantitative manner, with such functional properties of networks that are not evident in particular expressions.

Finally, Fig. 2F is a verbal description of the tensorial function implemented by a group of neuronal networks. The advantage of such concise definition as, for instance, "the cerebellum acts as a metric tensor of the motor hyperspace", is that it expresses in words a precise mathematical statement.

An additional advantage, provided by the use of tensor analysis, is that these abstract expressions can be made concrete at any of the various levels of abstraction, including the 'concrete' implementation by biological or man-made neuronal networks (cf. also Fig. 9).

Expansion of the concept of metric: tensorial interpretation of a sensorimotor system model

For the remainder of this paper, two-dimensional Euclidean illustrations (as in Fig. 2D) are used to broaden the application of the tensor concept to encompass several aspects of the sensorimotor system.